

# Functional versions of $L_p$ -affine surface area and entropy inequalities. \*

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## Abstract

In contemporary convex geometry, the rapidly developing  $L_p$ -Brunn Minkowski theory is a modern analogue of the classical Brunn Minkowski theory. A cornerstone of this theory is the  $L_p$ -affine surface area for convex bodies. Here, we introduce a functional form of this concept, for log concave and  $s$ -concave functions. We show that the new functional form is a generalization of the original  $L_p$ -affine surface area. We prove duality relations and affine isoperimetric inequalities for log concave and  $s$ -concave functions. This leads to a new inverse log-Sobolev inequality for  $s$ -concave densities.

## 1 Introduction.

The starting point of this paper is a reverse log-Sobolev inequality for log concave functions due to Artstein, Klartag, Schütt and Werner [3]. We first recall the usual log-Sobolev inequality. Let  $\gamma_n$  be the standard Gaussian measure on  $\mathbb{R}^n$ . The log-Sobolev inequality, due to Gross [18] (see also [14, 30]), asserts that for every probability measure  $\mu$  on  $\mathbb{R}^n$

$$H(\mu \mid \gamma_n) \leq \frac{1}{2} I(\mu \mid \gamma_n),$$

where  $H$  and  $I$  denote the relative entropy and Fisher information, respectively,

$$H(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} \log \left( \frac{d\mu}{d\gamma_n} \right) d\mu, \quad I(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} \left| \nabla \log \left( \frac{d\mu}{d\gamma_n} \right) \right|^2 d\mu$$

and  $|\cdot|$  is the Euclidean norm. It is well known (see for instance [5]) that this inequality can be slightly improved to

$$H(\mu \mid \gamma_n) \leq \frac{C(\mu)}{2} + \frac{n}{2} \log \left( 1 + \frac{I(\mu \mid \gamma_n) - C(\mu)}{n} \right), \quad (1)$$

where

$$C(\mu) = \int_{\mathbb{R}^n} |x|^2 d\mu - n$$

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\*Keywords: entropy, affine isoperimetric inequalities, log-Sobolev inequalities. 2010 Mathematics Subject Classification: 52A20.

<sup>†</sup>partially supported by the project GeMeCoD ANR 2011 BS01 007 01

<sup>‡</sup>Partially supported by an NSF grant

is the gap between the second moment of  $\mu$  and that of the Gaussian. The usual log-Sobolev inequality is recovered using the inequality  $\log(1+x) \leq x$ . Inequality (1) can be written in a more concise way. Put  $\psi = -\log(d\mu/dx)$  and let

$$S(\mu) = \int_{\mathbb{R}^n} \psi d\mu = -H(\mu \mid dx) = -H(\mu \mid \gamma_n) + \frac{C(\mu)}{2} + \frac{n}{2} \log(2\pi e)$$

be the Shannon entropy of  $\mu$ . Then  $S(\gamma_n) = \frac{n}{2} \log(2\pi e)$  so that

$$H(\mu \mid \gamma_n) - \frac{C(\mu)}{2} = S(\gamma_n) - S(\mu).$$

Moreover one has

$$I(\mu \mid \gamma_n) = \int |x - \nabla \psi(x)|^2 d\mu = C(\mu) + n + \int (|\nabla \psi(x)|^2 - 2\langle x, \nabla \psi(x) \rangle) d\mu.$$

Hence inequality (1) is equivalent to

$$2 \left( S(\gamma_n) - S(\mu) \right) \leq n \log \left( \frac{2n - 2 \int \langle x, \nabla \psi(x) \rangle d\mu + \int |\nabla \psi(x)|^2 d\mu}{n} \right).$$

If  $e^{-\psi}$  is  $C^2$  on  $\mathbb{R}^n$ , then  $\int \langle x, \nabla \psi(x) \rangle d\mu = n$  and  $\int |\nabla \psi(x)|^2 d\mu = \int \Delta \psi d\mu$  so that inequality (1) is equivalent to

$$2 \left( S(\gamma_n) - S(\mu) \right) \leq n \log \left( \frac{\int_{\mathbb{R}^n} \Delta \psi d\mu}{n} \right),$$

where  $\Delta$  is the Laplacian.

Recall that a measure  $\mu$  with density  $e^{-\psi}$  with respect to the Lebesgue measure is called log-concave if  $\psi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is a convex function. For such log-concave measures the following reversed form of the previous inequality holds. There,  $\nabla^2 \psi$  denotes the Hessian of  $\psi$ .

**Theorem 1.** *Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$ , with density  $e^{-\psi}$  with respect to the Lebesgue measure. Then*

$$\int_{\mathbb{R}^n} \log(\det(\nabla^2 \psi)) d\mu \leq 2 \left( S(\gamma_n) - S(\mu) \right).$$

*Equality holds if and only if  $\mu$  is Gaussian (with any mean and any positive definite covariance matrix).*

The inequality of Theorem 1 is due to Artstein, Klartag, Schütt and Werner [3], apart from the equality case which was left open and smoothness hypotheses which we removed. Their proof is based on affine isoperimetric inequalities and is pretty technical.

It is one aim of the present article to give a simple and short proof of this theorem including the characterization of equality, based on the functional form of the Blaschke-Santaló inequality.

This new approach can be extended to a more general scheme which we develop in subsequent sections. In particular, it leads to the definition of *functional  $L_p$ -affine surface area*. In Theorem 2 and Corollary 3, we establish, for log concave functions,

their corresponding duality relation and  $L_p$ -affine isoperimetric inequalities. Those are the counterparts to the ones that hold for convex bodies. In fact, we show that the  $L_p$ -affine isoperimetric inequalities for convex bodies can be obtained from the ones for log concave functions. This is explained in section 3.3..

Finally, we generalize the notion of  $L_p$ -affine isoperimetric surface area to  $s$ -concave functions for  $s > 0$ . We establish in Theorem 4 a duality relation which enables to prove the corresponding  $L_p$ -affine inequalities and the reverse log-Sobolev inequality for  $s$ -concave functions.

## 1.1 Notations

For a convex function  $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , we define  $\Omega_\psi$  to be the interior of the convex domain of  $\psi$ ,  $\{x \in \mathbb{R}^n, \psi(x) < +\infty\}$ . We always consider in this paper convex functions  $\psi$  such that  $\Omega_\psi \neq \emptyset$ . We will use the classical Legendre transform of  $\psi$ ,

$$\psi^*(y) = \sup_x (\langle x, y \rangle - \psi(x)). \quad (2)$$

In the general case, when  $\psi$  is neither smooth nor strictly convex, the gradient of  $\psi$ , denoted by  $\nabla\psi$ , exists almost everywhere by Rademacher's theorem (e.g., [8]), and a theorem of Alexandrov [1] and Busemann and Feller [9] guarantees the existence of its Hessian, denoted  $\nabla^2\psi$ , almost everywhere in  $\Omega_\psi$ . We let  $X_\psi$  be the set of points of  $\Omega_\psi$  at which its Hessian  $\nabla^2\psi$  in the sense of Alexandrov exists and is invertible. Recall also that

$$\psi(x) + \psi^*(y) \geq \langle x, y \rangle$$

for every  $x, y \in \mathbb{R}^n$ , with equality if and only if  $x$  is in the domain of  $\psi$  and  $y \in \partial\psi(x)$ , the sub differential of  $\psi$  at  $x$ . In particular

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x) \rangle - \psi(x), \quad \text{a.e. in } \Omega_\psi. \quad (3)$$

References about duality of convex functions are [26, 27, 28]. We will denote by  $|x|$  the Euclidean norm of a vector  $x$  in  $\mathbb{R}^n$ .

## 2 A short proof of the reverse log-Sobolev inequality

Let us first recall the form of the functional Blaschke-Santaló inequality [2, 6, 16, 21] that we need. Let  $f, g$  be non-negative integrable functions on  $\mathbb{R}^n$  satisfying

$$f(x)g(y) \leq e^{-\langle x, y \rangle}, \quad \forall x, y \in \mathbb{R}^n.$$

If  $f$  has its barycenter at 0, which means that  $\int x f(x) dx = 0$ , then

$$\left( \int_{\mathbb{R}^n} f dx \right) \times \left( \int_{\mathbb{R}^n} g dx \right) \leq (2\pi)^n.$$

There is equality if and only if there exists a positive definite matrix  $A$  and  $C > 0$  such that, a.e. in  $\mathbb{R}^n$ ,

$$f(x) = C e^{-\langle Ax, x \rangle / 2}, \quad g(y) = \frac{e^{-\langle A^{-1}y, y \rangle / 2}}{C}.$$

*Proof of Theorem 1.* Without loss of generality, we may assume that the function  $\psi$  is lower semi-continuous. Both terms of the inequality are invariant under translations of the measure  $\mu$ , so we can assume that  $\mu$  has its barycenter at 0. Then by the functional Santaló inequality above

$$\int_{\mathbb{R}^n} e^{-\psi^*} dx \leq (2\pi)^n. \quad (4)$$

Let  $\Omega_\psi, \Omega_{\psi^*}$  be the interiors of the domains of  $\psi$  and  $\psi^*$ , respectively. If  $\psi$  is  $\mathcal{C}^2$ -smooth and strictly convex then the map  $\nabla\psi: \Omega_\psi \rightarrow \Omega_{\psi^*}$  is smooth and bijective. So by the change of variable formula,

$$\int_{\mathbb{R}^n} e^{-\psi^*(y)} dy = \int_{\Omega_{\psi^*}} e^{-\psi^*(y)} dy = \int_{\Omega_\psi} e^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x)) dx. \quad (5)$$

As noted above, in the general case, Rademacher's theorem still guarantees the existence of the gradient  $\nabla\psi$  of  $\psi$  and a theorem of Alexandrov and Busemann and Feller the existence of its Hessian  $\nabla^2\psi$ , almost everywhere in  $\Omega$ , so that both terms of equality (5) make sense. Although it is clear (take  $\psi(x) = |x|$  in  $\mathbb{R}$ ) that this equality may fail in general, a result of McCann [26, Corollary 4.3 and Proposition A.1] shows that

$$\int_{\Omega_\psi} e^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x)) dx = \int_{X_{\psi^*}} e^{-\psi^*(y)} dy, \quad (6)$$

where  $X_{\psi^*}$  is the set of vectors of  $\Omega_{\psi^*}$  at which  $\nabla^2\psi^*$  exists and is invertible. Together with (4) we get

$$\int_{\Omega_\psi} e^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x)) dx \leq (2\pi)^n.$$

With (3), the previous inequality thus becomes

$$\int_{\Omega_\psi} e^{-\langle x, \nabla\psi(x) \rangle + \psi(x)} \det(\nabla^2\psi(x)) dx \leq (2\pi)^n,$$

which can be rewritten as

$$\int_{\mathbb{R}^n} e^{-\langle x, \nabla\psi(x) \rangle + 2\psi(x)} \det(\nabla^2\psi(x)) d\mu \leq (2\pi)^n. \quad (7)$$

Taking the logarithm and using Jensen's inequality (recall that  $\mu$  is assumed to be a probability measure) we obtain

$$-\int_{\mathbb{R}^n} \langle x, \nabla\psi(x) \rangle d\mu + 2S(\mu) + \int_{\mathbb{R}^n} \log(\det(\nabla^2\psi)) d\mu \leq n \log(2\pi).$$

We will need some version of the Gauss-Green (or Stokes) formula and refer to [12] for general references and recent results on this subject. Let  $v$  be the vector flow  $v(x) = e^{-\psi(x)}x$ . By convexity and lower semi-continuity of  $\psi$ , it is continuous and locally Lipschitz on  $\overline{\Omega_\psi}$ . Assume first that  $\Omega_\psi$  is bounded. Then by the Gauss-Green formula [13, 15], we have

$$\int_{\Omega_\psi} \operatorname{div}(v(x)) dx = \int_{\partial\Omega_\psi} \langle v(x), N_{\Omega_\psi}(x) \rangle d\sigma_{\Omega_\psi},$$

where  $N_{\Omega_\psi}(x)$  is an exterior normal to the convex set  $\Omega_\psi$  at the point  $x$  and  $\sigma_{\Omega_\psi}$  is the surface area measure on  $\partial\Omega_\psi$ . Hence

$$\begin{aligned} \int_{\mathbb{R}^n} \langle x, \nabla \psi(x) \rangle d\mu &= \int_{\Omega_\psi} \langle x, \nabla \psi(x) \rangle e^{-\psi(x)} dx \\ &= \int_{\Omega_\psi} \operatorname{div}(x) e^{-\psi(x)} dx - \int_{\partial\Omega_\psi} \langle x, N_{\Omega_\psi}(x) \rangle e^{-\psi(x)} d\sigma_{\Omega_\psi}. \end{aligned}$$

This formula holds true for unbounded domain  $\Omega_\psi$  by a simple truncation argument and by the fast decay of log-concave integrable functions. Since  $\Omega_\psi$  is convex, the barycenter 0 of  $\mu$  is in  $\Omega_\psi$ . Thus  $\langle x, N_{\Omega_\psi}(x) \rangle \geq 0$  for every  $x \in \partial\Omega_\psi$  and  $\operatorname{div}(x) = n$  hence

$$\int_{\mathbb{R}^n} \langle x, \nabla \psi(x) \rangle d\mu \leq n.$$

This finishes the proof of the inequality. Let us move on to the equality case. It is easily checked that there is equality in Theorem 1 for Gaussian measures. On the other hand, the above proof shows that if  $\mu$  satisfies the equality case, then there must be equality in (4). Then, by the equality case of the functional Santaló inequality,  $\mu$  is Gaussian.  $\square$

### 3 A functional $L_p$ -affine surface area.

#### 3.1 General theorems.

We first present a definition that generalizes the notion of  $L_p$ -affine surface area of convex bodies to a functional setting. Generalizations of a different nature were given in [10] and [11].

**Definition 1.** For  $F_1, F_2: \mathbb{R} \rightarrow (0, +\infty)$  and  $\lambda \in \mathbb{R}$ , we define

$$as_\lambda(F_1, F_2, \psi) = \int_{X_\psi} \left( F_1(\psi(x)) \right)^{1-\lambda} \left( F_2(\langle x, \nabla \psi(x) \rangle - \psi(x)) \right)^\lambda \left( \det \nabla^2 \psi(x) \right)^\lambda dx. \quad (8)$$

Since  $\det(\nabla^2 \psi(x)) = 0$  outside  $X_\psi$ , the integral may be taken on  $\Omega_\psi$  for  $\lambda > 0$ . Definition 1 is motivated by two important facts. Firstly, we can prove that for a particular choice of  $F_1$ ,  $F_2$  and  $\psi$  it fits with the usual  $L_p$ -affine surface area of a convex body. This is the content of Theorem 3. Secondly, in the case of log-concave functions, for  $F_1(t) = F_2(t) = e^{-t}$  the functional affine surface area  $as_1(F_1, F_2, \psi)$  becomes

$$as_1(F_1, F_2, \psi) = \int_{X_\psi} e^{-\psi^*(\nabla \psi(x))} \det \nabla^2 \psi(x) dx = \int_{\Omega_\psi} e^{-\psi^*(\nabla \psi(x))} \det \nabla^2 \psi(x) dx$$

and is of particular interest. This is illustrated in subsection 3.2.

Our main result is the duality formula of Theorem 2. A special case is the identity (6) which was the starting point of the short proof of the reverse log-Sobolev inequality presented in the Section 2.

Notice also that for any linear invertible map  $A$  on  $\mathbb{R}^n$ , one has

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda-1} as_\lambda(F_1, F_2, \psi), \quad (9)$$

which corresponds to an  $SL(n)$  invariance with a homogeneity of degree  $(2\lambda - 1)$ . This is easily checked using that  $\nabla_x(\psi \circ A) = A^t \nabla_{Ax} \psi$  and  $\nabla_x^2(\psi \circ A) = A^t \nabla_{Ax}^2 \psi A$ .

We shall use Corollary 4.3 and Proposition A.1 of [26], where McCann showed a general change of variable formula, namely for every Borel function  $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,

$$\int_{X_\psi} f(\nabla \psi(x)) \det \nabla^2 \psi(x) dx = \int_{X_{\psi^*}} f(y) dy. \quad (10)$$

The same holds true for every integrable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Identity (10) is obvious when  $\psi$  satisfies some regularity assumptions, like  $C^2$ . It suffices to make the change of variable  $y = \nabla \psi(x)$ . The proofs are however more delicate in a general setting.

We establish the following duality relation.

**Theorem 2.** *Let  $\lambda \in \mathbb{R}$ , let  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}_+$  and let  $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex. If  $\lambda < 0$  or  $\lambda > 1$ , assume moreover that  $F_1 \circ \psi > 0$  on  $X_\psi$  and  $F_2 \circ \psi^* > 0$  on  $X_{\psi^*}$ . Then*

$$as_\lambda(F_1, F_2, \psi) = as_{1-\lambda}(F_2, F_1, \psi^*).$$

*Proof.* Without loss of generality, we can assume that  $\psi$  is lower semi-continuous so that  $\psi = (\psi^*)^*$ . By (3),

$$as_\lambda(F_1, F_2, \psi) = \int_{X_\psi} (F_1 \circ \psi(x))^{1-\lambda} (F_2 \circ \psi^*(\nabla \psi(x)))^\lambda (\det \nabla^2 \psi(x))^\lambda dx.$$

By Proposition A.1 in [26],

$$x = \nabla \psi^* \circ \nabla \psi(x) \quad \text{and} \quad \nabla^2 \psi^*(\nabla \psi(x)) = (\nabla^2 \psi(x))^{-1}, \quad \forall x \in X_\psi,$$

so that  $as_\lambda(F_1, F_2, \psi)$  is equal to

$$\int_{X_\psi} (F_1 \circ \psi \circ \nabla \psi^*(\nabla \psi(x)))^{1-\lambda} (F_2 \circ \psi^*(\nabla \psi(x)))^\lambda (\det \nabla^2 \psi^*(\nabla \psi(x)))^{1-\lambda} \det \nabla^2 \psi(x) dx.$$

With (10), we get that

$$as_\lambda(F_1, F_2, \psi) = \int_{X_{\psi^*}} (F_1 \circ \psi \circ \nabla \psi^*(y))^{1-\lambda} (F_2 \circ \psi^*(y))^\lambda (\det \nabla^2 \psi^*(y))^{1-\lambda} dy.$$

We conclude the proof using (3) with  $\psi^*$  and  $(\psi^*)^* = \psi$ . □

**Corollary 1.** *The function  $\lambda \mapsto \log(as_\lambda(F_1, F_2, \psi))$  is convex on  $\mathbb{R}$ . Moreover,*

$$\forall \lambda \in [0, 1], \quad as_\lambda(F_1, F_2, \psi) \leq \left( \int_{X_\psi} F_1 \circ \psi \right)^{1-\lambda} \left( \int_{X_{\psi^*}} F_2 \circ \psi^* \right)^\lambda.$$

*Equality holds trivially if  $\lambda = 0$  and  $\lambda = 1$ .*

$$\forall \lambda \notin [0, 1], \quad as_\lambda(F_1, F_2, \psi) \geq \left( \int_{X_\psi} F_1 \circ \psi \right)^{1-\lambda} \left( \int_{X_{\psi^*}} F_2 \circ \psi^* \right)^\lambda.$$

*Proof.* The convexity of  $\lambda \mapsto \log(as_\lambda(F_1, F_2, \psi))$  is a consequence of Hölder inequality. For the inequalities we use Hölder inequality and also the duality relation of Theorem 2 with  $\lambda = 1$ ,  $as_1(F_1, F_2, \psi) = as_0(F_2, F_1, \psi^*) = \int_{X_{\psi^*}} F_2 \circ \psi^*$ .  $\square$

We define the non-increasing function  $F : \mathbb{R} \rightarrow \mathbb{R}_+$  by

$$F(t) = \sup_{\frac{t_1+t_2}{2} \geq t} \sqrt{F_1(t_1)F_2(t_2)}. \quad (11)$$

Notice that if  $F_1 = F_2$  is a log-concave, non-increasing function then  $F = F_1 = F_2$ .

**Corollary 2.** *Let  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}_+$ , let  $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. Then there exists  $z \in \mathbb{R}^n$  such that*

$$\forall \lambda \in [0, 1/2], \quad as_\lambda(F_1, F_2, \psi_z) \leq \left( \int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx \right)^{2\lambda} \left( \int_{X_\psi} F_1 \circ \psi \right)^{1-2\lambda}$$

*Equality holds trivially if  $\lambda = 0$ . If  $F_1 \circ \psi > 0$  on  $X_\psi$  and  $F_2 \circ \psi^* > 0$  on  $X_{\psi^*}$  then*

$$\forall \lambda < 0, \quad as_\lambda(F_1, F_2, \psi_z) \geq \left( \int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx \right)^{2\lambda} \left( \int_{X_\psi} F_1 \circ \psi \right)^{1-2\lambda},$$

where  $\psi_z(x) = \psi(z + x)$ .

*If  $F$  is decreasing,  $\lambda \neq 0$  and  $\int_{X_\psi} F_1 \circ \psi \neq 0$ , then there is equality in each of these inequalities if and only if there exists  $c \in \mathbb{R}_+$ ,  $a \in \mathbb{R}$  and a positive definite matrix  $A$  such that, for every  $x \in \mathbb{R}^n$  and  $t \geq 0$ ,*

$$\psi_z(x) = \langle Ax, x \rangle + a, \quad F_1(t + a) = c F(t) \quad \text{and} \quad F_2(t - a) = \frac{F(t)}{c}.$$

**Remark.** (i) Notice that if  $\psi$  is even then one may choose  $z = 0$ .

(ii) Moreover, for  $\lambda > 1/2$ , we deduce from the duality relation proved in Theorem 2 that the same inequalities hold true exchanging  $F_1$  and  $F_2$ ,  $\psi$  and  $\psi^*$  and that the equality case is characterized for  $\lambda \neq 1$ .

*Proof.* We recall a general form of the functional Blaschke-Santaló inequality [16, 22]. Let  $f$  be a non-negative integrable function on  $\mathbb{R}^n$ . There exists  $z_0 \in \mathbb{R}^n$  such that for every  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and every  $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$  satisfying

$$f(z_0 + x)g(y) \leq (\rho(\langle x, y \rangle))^2, \quad (12)$$

for every  $x, y \in \mathbb{R}^n$  with  $\langle x, y \rangle > 0$ , we have

$$\int_{\mathbb{R}^n} f dx \int_{\mathbb{R}^n} g dx \leq \left( \int_{\mathbb{R}^n} \rho(|x|^2) dx \right)^2. \quad (13)$$

If  $f$  is even, a result of Ball [6] asserts that one may choose  $z_0 = 0$ . Moreover, if there exists  $g$  satisfying (12) and equality holds in (13), then there exists  $c > 0$  and an invertible  $T$ , such that for every  $x \in \mathbb{R}^n$ ,

$$f(z_0 + x) = c\rho(|Tx|^2) \quad \text{and} \quad g(y) = \frac{1}{c}\rho(|T^{-1}x|^2). \quad (14)$$

For  $z \in \mathbb{R}^n$ , let us denote  $\psi_z^* = (\psi_z)^*$ . Since  $F$  is non-increasing, we have by (2), for every  $x, y, z \in \mathbb{R}^n$  such that  $\langle x, y \rangle > 0$ ,

$$F_1(\psi_z(x))F_2(\psi_z^*(y)) \leq F^2\left(\frac{\psi_z(x) + \psi_z^*(y)}{2}\right) \leq F^2\left(\frac{\langle x, y \rangle}{2}\right).$$

By the functional Blaschke-Santaló inequality there exists  $z_0 \in \mathbb{R}^n$  such that

$$\left(\int F_1 \circ \psi\right) \left(\int F_2 \circ \psi_{z_0}^*\right) \leq \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^2. \quad (15)$$

Applying Corollary 1 to  $\psi_{z_0}$ , we deduce that for  $\lambda \in [0, 1]$ ,

$$\begin{aligned} as_\lambda(F_1, F_2, \psi_{z_0}) &\leq \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-\lambda} \left(\int_{X_{\psi_{z_0}^*}} F_2 \circ \psi_{z_0}^*\right)^\lambda \\ &\leq \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda} \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-2\lambda}. \end{aligned}$$

For  $\lambda < 0$  we deduce from (15) that

$$\left(\int_{X_\psi} F_1 \circ \psi\right)^\lambda \left(\int_{X_{\psi_{z_0}^*}} F_2 \circ \psi_{z_0}^*\right)^\lambda \geq \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda}$$

and we conclude by using the second part of Corollary 1.

To characterize the equality case, we suppose that  $\int_{X_\psi} F_1 \circ \psi \neq 0$  which means that the expressions are not identically zero in the inequality. For  $\lambda \neq 0$ , if there is equality in one of the inequalities of Corollary 2, it follows from the proof that we have equality in the functional Blaschke-Santaló inequality. Thus by (14), there exists  $c > 0$  and an invertible matrix  $T$ , such that for every  $x \in \mathbb{R}^n$ ,

$$F_1 \circ \psi_{z_0}(x) = c F\left(\frac{|Tx|^2}{2}\right) \quad \text{and} \quad F_2 \circ \psi_{z_0}^*(x) = \frac{1}{c} F\left(\frac{|T^{-1}x|^2}{2}\right).$$

Let us define  $\varphi(x) = \psi(T^{-1}x + z_0)$ . Then we have

$$F_1 \circ \varphi(x) = c F\left(\frac{|x|^2}{2}\right) \quad \text{and} \quad F_2 \circ \varphi^*(x) = \frac{1}{c} F\left(\frac{|x|^2}{2}\right). \quad (16)$$

Hence

$$F\left(\frac{|x|^2}{2}\right) = \sqrt{F_1 \circ \varphi(x) F_2 \circ \varphi^*(x)} \leq F\left(\frac{\varphi(x) + \varphi^*(x)}{2}\right) \leq F\left(\frac{|x|^2}{2}\right).$$

Since  $F$  is decreasing, we deduce that  $\varphi(x) + \varphi^*(x) = |x|^2$ . It is classical that this implies that  $\varphi(x) = |x|^2/2 + a$ . See for example the argument given in the proof of Theorem 8 in [16]. Defining  $A = T^*T/2$ , we get that  $\psi_{z_0}(x) = \langle Ax, x \rangle + a$ , for every  $x \in \mathbb{R}^n$ . From (16) we deduce that for every  $t \geq 0$

$$F_1(t + a) = c F(t) \quad \text{and} \quad F_2(t - a) = \frac{1}{c} F(t).$$

Therefore all the conditions of the theorem are proved. Reciprocally, if these conditions are fulfilled, a simple computation shows that there is equality.  $\square$



### 3.2 Application to particular functions: the log-concave case.

We define  $F_1$  and  $F_2$  on  $\mathbb{R}$  by  $F_1(t) = F_2(t) = e^{-t}$ ; then  $F(t) = e^{-t}$  as well and we use the simplified notation

$$as_\lambda(\psi) = as_\lambda(e^{-t}, e^{-t}, \psi) = \int_{X_\psi} e^{(2\lambda-1)\psi(x) - \lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2 \psi(x))^\lambda dx. \quad (17)$$

Again, as before, we can replace  $X_\psi$  by  $\Omega_\psi$  for  $\lambda > 0$ . Observe that for the Euclidean norm  $|\cdot|$ ,

$$as_\lambda\left(\frac{|\cdot|^2}{2}\right) = (2\pi)^{\frac{n}{2}}. \quad (18)$$

Moreover, it is not difficult to see (see e.g., [10]) that for any  $\lambda \in \mathbb{R}$ ,  $\psi \mapsto as_\lambda(\psi)$  is a valuation on the set of convex functions  $\psi$ , i.e., if  $\min(\psi_1, \psi_2)$  is convex, then

$$as_\lambda(\psi_1) + as_\lambda(\psi_2) = as_\lambda(\max(\psi_1, \psi_2)) + as_\lambda(\min(\psi_1, \psi_2)),$$

and it is homogeneous of degree  $(2\lambda - 1)n$ , since we have by (9) for any linear invertible map  $A$  on  $\mathbb{R}^n$ , for all convex  $\psi$

$$as_\lambda(\psi \circ A) = |\det A|^{2\lambda-1} as_\lambda(\psi).$$

For convex bodies with the origin in their interiors, such upper semi-continuous valuations were characterized as  $L_p$ -affine surface areas in [23] and [24] which motivated us to call  $as_\lambda(\psi)$  the  $L_\lambda$ -affine surface area of  $\psi$ . This is further justified by Theorem 3 of the next section (where we also give the definition of  $L_p$ -affine surface area for convex bodies), and by the identity (26) of Section 4.

From Theorem 2 and Corollary 1 we get that  $\lambda \mapsto \log(as_\lambda(\psi))$  is convex and that

$$\forall \lambda \in \mathbb{R}, \quad as_\lambda(\psi) = as_{1-\lambda}(\psi^*). \quad (19)$$

The following isoperimetric inequalities are a direct consequence of Corollary 2 and a result of [22] which says that the Santaló point  $z_0$  in the functional Blaschke-Santaló inequality (12) can be taken equal to 0 when  $\int x e^{-\psi(x)} dx = 0$  or  $\int x e^{-\psi^*(x)} dx = 0$ .

**Corollary 3.** *Let  $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function such that  $\int x e^{-\psi(x)} dx = 0$  or  $\int x e^{-\psi^*(x)} dx = 0$ . Then*

$$\begin{aligned} \forall \lambda \in [0, 1/2], \quad as_\lambda(\psi) &\leq (2\pi)^{n\lambda} \left( \int_{X_\psi} e^{-\psi} \right)^{1-2\lambda}, \\ \forall \lambda \in (-\infty, 0], \quad as_\lambda(\psi) &\geq (2\pi)^{n\lambda} \left( \int_{X_\psi} e^{-\psi} \right)^{1-2\lambda}. \end{aligned}$$

*Equality holds in both inequalities for  $\lambda \neq 0$ , if and only if there exists  $a \in \mathbb{R}$  and a positive definite matrix  $A$  such that  $\psi(x) = \langle Ax, x \rangle + a$ , for every  $x \in \mathbb{R}^n$ .*

**Remark.** (i) *To emphasize the isoperimetric character of these inequalities, note that with (18), the inequalities are equivalent to*

$$\forall \lambda \in [0, 1/2], \quad \frac{as_\lambda(\psi)}{as_\lambda\left(\frac{|\cdot|^2}{2}\right)} \leq \left( \frac{\int_{X_\psi} e^{-\psi}}{\int e^{-\frac{|\cdot|^2}{2}}} \right)^{1-2\lambda}$$

and

$$\forall \lambda < 0, \quad \frac{as_\lambda(\psi)}{as_\lambda\left(\frac{|\cdot|^2}{2}\right)} \geq \left(\frac{\int_{X_\psi} e^{-\psi}}{\int e^{-\frac{|\cdot|^2}{2}}}\right)^{1-2\lambda}.$$

(ii) It follows from Corollary 3 and the functional Blaschke Santaló inequality that

$$\forall \lambda \in [0, 1/2], \quad as_\lambda(\psi)as_\lambda(\psi^*) \leq (2\pi)^n.$$

There are several other direct consequences of Corollary 3 that should be noticed. As observed already, we have for every  $\lambda \in (0, 1/2]$ ,

$$as_\lambda(\psi) = \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx.$$

Since  $\int_{X_\psi} e^{-\psi} \leq \int e^{-\psi}$  we deduce from Corollary 3 that for any  $\lambda \in (0, 1/2]$ ,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx \leq (2\pi)^{n\lambda} \left(\int e^{-\psi}\right)^{1-2\lambda}. \quad (20)$$

This inequality holds trivially true also for  $\lambda = 0$ . Moreover, by Theorem 2, we know that  $as_\lambda(\psi) = as_{1-\lambda}(\psi^*)$ . Since the inequalities of Corollary 3 are also valid when  $\int xe^{-\psi^*(x)}dx = 0$ , we deduce from (20) that if  $\lambda \in [1/2, 1]$ ,

$$\begin{aligned} \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx &= as_\lambda(\psi) \\ &= as_{1-\lambda}(\psi^*) \leq (2\pi)^{n(1-\lambda)} \left(\int e^{-\psi^*}\right)^{2\lambda-1}. \end{aligned}$$

By the Blaschke-Santaló functional inequality (see (15)), we know that  $\int e^{-\psi} \int e^{-\psi^*} \leq (2\pi)^n$  and we conclude that for all  $\lambda \in [1/2, 1]$ ,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx \leq (2\pi)^{n\lambda} \left(\int e^{-\psi}\right)^{1-2\lambda}.$$

For  $\lambda < 0$  or  $\lambda > 1$ , an important case concerns  $C^2$  convex functions  $\psi$ . In such a situation  $X_\psi = \Omega_\psi$  and  $X_{\psi^*} = \Omega_{\psi^*}$  and we deduce from Corollary 2 that for all  $\lambda < 0$ ,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx \geq (2\pi)^{n\lambda} \left(\int e^{-\psi}\right)^{1-2\lambda}.$$

For all  $\lambda > 1$ , we go back to Corollary 1 and deduce that

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx = as_\lambda(\psi) \geq \left(\int e^{-\psi}\right)^{1-\lambda} \left(\int e^{-\psi^*}\right)^\lambda.$$

By the asymptotic functional reverse Santaló inequality [17] (see also [20] in the even case), there exists a constant  $c > 0$  such that  $\int e^{-\psi} \int e^{-\psi^*} \geq c^n$ . Therefore, for all  $\lambda > 1$ ,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x, \nabla\psi(x) \rangle} (\det \nabla^2\psi(x))^\lambda dx \geq c^{n\lambda} \left(\int e^{-\psi}\right)^{1-2\lambda}.$$

We have proved

**Corollary 4.** *Let  $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function such that  $\int x e^{-\psi(x)} dx = 0$  or  $\int x e^{-\psi^*(x)} dx = 0$ . Then*

$$\forall \lambda \in [0, 1], \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x) - \lambda \langle x, \nabla \psi(x) \rangle} (\det \nabla^2 \psi(x))^\lambda dx \leq (2\pi)^{n\lambda} \left( \int e^{-\psi} \right)^{1-2\lambda},$$

Moreover, if  $\psi \in C^2(\Omega_\psi)$ ,

$$\forall \lambda < 0, \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x) - \lambda \langle x, \nabla \psi(x) \rangle} (\det \nabla^2 \psi(x))^\lambda dx \geq (2\pi)^{n\lambda} \left( \int e^{-\psi} \right)^{1-2\lambda}$$

and there exists an absolute constant  $c > 0$  such that

$$\forall \lambda > 1, \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x) - \lambda \langle x, \nabla \psi(x) \rangle} (\det \nabla^2 \psi(x))^\lambda dx \geq c^{n\lambda} \left( \int e^{-\psi} \right)^{1-2\lambda}.$$

These are the complete analogues of the  $L_p$ -affine surface area inequalities due to [25, 19, 29] and this will be discussed in more details in the next subsection.

### 3.3 The case of convex bodies.

We continue to study the case  $F_1(t) = F_2(t) = e^{-t}$ . Additionally, we consider the case of 2-homogeneous proper convex functions  $\psi$ , that is  $\psi(\lambda x) = \lambda^2 \psi(x)$  for any  $\lambda \in \mathbb{R}_+$  and  $x \in \mathbb{R}^n$ . Such functions  $\psi$  are necessarily (and this is obviously sufficient) of the form  $\psi(x) = \|x\|_K^2/2$  for a certain convex body  $K$  with 0 in its interior. Here,  $\|\cdot\|_K$  is the gauge function the convex body  $K$ ,

$$\|x\|_K = \min\{\alpha \geq 0 : x \in \alpha K\} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x).$$

Differentiating with respect to  $\lambda$  at  $\lambda = 1$ , we get

$$\langle x, \nabla \psi(x) \rangle = 2\psi(x).$$

Thus for 2-homogeneous functions  $\psi$ , formula (17) further simplifies to

$$as_\lambda(\psi) = \int_{X_\psi} (\det \nabla^2 \psi(x))^\lambda e^{-\psi(x)} dx, \quad (21)$$

where  $X_\psi$  is the positive cone generated by the points of  $\partial K$  where the Gauss curvature is strictly positive. The following theorem indicates why we call  $as_\lambda(\psi)$  the  $L_\lambda$ -affine surface area of  $\psi$ . First we recall that for  $p \in \mathbb{R}$ ,  $p \neq -n$ , the  $L_p$ -affine surface area for a convex body  $K$  in  $\mathbb{R}^n$  with the origin in its interior is defined [19, 25, 29] as

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x). \quad (22)$$

Here,  $N_K(x)$  is the outer unit normal to the boundary  $\partial K$  in the boundary point  $x$ ,  $\mu_K$  is the usual surface area measure on  $\partial K$  and  $\kappa_K(x)$  is the Gauss curvature in  $x$ . We denote by  $(\partial K)_+$  the points of  $\partial K$  where the Gauss curvature is strictly positive.

**Theorem 3.** Let  $K$  be a convex body in  $\mathbb{R}^n$  containing the origin in its interior. For any  $p \geq 0$ , let  $\lambda = \frac{p}{n+p}$ . Then

$$as_\lambda \left( \frac{\|\cdot\|_K^2}{2} \right) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} as_p(K).$$

Moreover, if  $(\partial K)_+$  has full Lebesgue measure in  $\partial K$ , then the same relation holds true for every  $p \neq -n$ .

**Remark.** For all  $p$ ,  $as_p(B_2^n) = n|B_2^n|$ . Therefore, together with (18), the identity given in the theorem can be written as

$$\frac{as_\lambda \left( \frac{\|\cdot\|_K^2}{2} \right)}{as_\lambda \left( \frac{\|\cdot\|^2}{2} \right)} = \frac{as_p(K)}{as_p(B_2^n)}.$$

We will need the following technical lemma.

**Lemma 1.** Let  $K$  be a convex body in  $\mathbb{R}^n$  with the origin in its interior and let  $\psi(x) = \frac{1}{2}\|x\|_K^2$ . Then for all  $x \in (\partial K)_+$ ,

$$\det(\nabla^2 \psi(x)) = \frac{\kappa_K(x)}{\|G_K(x)\|_{K^\circ}^{n+1}},$$

where  $G_K: (\partial K)_+ \rightarrow \mathbb{S}^{n-1}$  is the Gauss map.

*Proof.* Let us fix  $x \in (\partial K)_+$ . The differential  $d_x G_K$  of  $G_K$  at  $x \in \partial K$  is a linear map from the tangent space  $T_x(\partial K)$  to  $T_{G_K(x)}(\mathbb{S}^{n-1})$ . We can identify both spaces with  $G_K(x)^\perp$  and view  $d_x G_K$  as a linear operator on  $G_K(x)^\perp$ . Then by definition (see e.g., [28])

$$\kappa_K(x) = \det(d_x G_K(x)).$$

Let  $f: x \in \mathbb{R}^n \mapsto \|x\|_K$ . For all  $x \neq 0$ , consider  $N_K$ , the 0-homogeneous extension of  $G_K$ , defined by  $N_K(x) = G_K\left(\frac{x}{\|x\|_K}\right)$ . Then

$$\nabla f(x) = \frac{N_K(x)}{\|N_K(x)\|_{K^\circ}}.$$

Using the identity

$$N_{K^\circ}(N_K(x)) = \frac{x}{\|x\|_2},$$

we get

$$\nabla^2 f(x) = \frac{d_x N_K}{\|N_K(x)\|_{K^\circ}} - \frac{((d_x N_K)^T x) \otimes N_K(x)}{\|N_K(x)\|_{K^\circ}^2}.$$

Therefore, if we put  $A = d_x N_K$ ,  $u = N_K(x)$  and  $a = \|N_K(x)\|_{K^\circ}$ ,

$$\nabla^2 \psi(x) = \nabla^2 (f^2(x)/2) = \frac{A}{a} - \frac{(A^T x) \otimes u}{a^2} + \frac{u \otimes u}{a^2}.$$

Let  $B = d_x G_K$ . Then  $Ay = By$  for every  $y \in u^\perp$ . Since  $N_K$  is 0-homogeneous,  $Ax = 0$ . Thus

$$Au = -\frac{Bx_\perp}{\langle x, u \rangle} = -\frac{Bx_\perp}{a}$$

where  $x_\perp = x - \langle x, u \rangle u \in u^\perp$ . Also, as  $B$  is self-adjoint (see e.g. [28]),

$$\begin{aligned}\langle A^T x, y \rangle &= \langle x, By \rangle = \langle Bx_\perp, y \rangle, \quad \forall y \in u^\perp \\ \langle A^T x, u \rangle &= -\frac{1}{a} \langle x, Bx_\perp \rangle = -\frac{1}{a} \langle x_\perp, Bx_\perp \rangle.\end{aligned}$$

The previous computations show that in a basis adapted to the decomposition  $\mathbb{R}^n = \text{span}(u) + u^\perp$ , we have

$$\nabla^2 \psi(x) = \frac{1}{a^3} \begin{bmatrix} a + \langle x_\perp, Bx_\perp \rangle & -a(Bx_\perp)^T \\ -aBx_\perp & a^2 B \end{bmatrix}.$$

Observe that

$$\nabla^2 \psi(x) = \frac{1}{a^3} \begin{bmatrix} a & -(Bx_\perp)^T \\ 0 & aB \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -x_\perp & a \text{id}_{n-1} \end{bmatrix}.$$

Therefore,

$$\det(\nabla^2 \psi(x)) = a^{-n-1} \det(B) = \frac{\kappa_K(x)}{\|N_K(x)\|_{K^\circ}^{n+1}},$$

which is the result.  $\square$

*Proof of Theorem 3.* We will use formula (21) for  $\psi = \frac{\|\cdot\|_K^2}{2}$  and integrate in polar coordinates with respect to the normalized cone measure  $\sigma_K$  of  $K$ . Thus, if we write  $x = r\theta$ , with  $\theta \in \partial K$ ,  $dx = n|K|r^{n-1}drd\sigma_K(\theta)$ . We also use that the map  $x \mapsto \det \nabla^2 \psi(x)$  is 0-homogeneous. Therefore we get with (21),

$$\begin{aligned}as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) &= n|K| \int_0^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr \int_{(\partial K)_+} (\det \nabla^2 \psi(\theta))^\lambda d\sigma_K(\theta) \\ &= (2\pi)^{\frac{n}{2}} \frac{|K|}{|B_2^n|} \int_{(\partial K)_+} (\det \nabla^2 \psi(\theta))^\lambda d\sigma_K(\theta).\end{aligned}$$

The relation between the normalized cone measure  $\sigma_K$  and the Hausdorff measure  $\mu_K$  on  $\partial K$  is given by

$$d\sigma_K(x) = \frac{\langle x, N_K(x) \rangle d\mu_K(x)}{n|K|}.$$

Observe that for the function  $G_K(x)$  introduced in Lemma 1,  $\|G_K(x)\|_{K^\circ} = \langle x, N_K(x) \rangle$ . Thus, with  $\lambda = \frac{p}{n+p}$ ,

$$\begin{aligned}as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} \int_{(\partial K)_+} \left( \frac{\kappa(x)}{\langle x, N_K(x) \rangle^{n+1}} \right)^\lambda \langle x, N_K(x) \rangle d\mu_K(x) \\ &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} as_p(K),\end{aligned}$$

when  $\lambda \in [0, 1)$  or when  $(\partial K)_+$  is of full Lebesgue measure in  $\partial K$ .  $\square$

Let us conclude this section with several observations. First, observe that

$$\int e^{-\frac{\|x\|_K^2}{2}} dx = 2^{\frac{n}{2}} \Gamma\left(1 + \frac{n}{2}\right) |K|.$$

Combining this with Theorem 3 and Corollary 3, we recover the known  $L_p$ -affine isoperimetric inequalities for convex bodies. Namely, for a convex body  $K$  with the origin in its interior, we get for  $\lambda \in [0, 1)$ , which corresponds to  $p \in [0, \infty)$  ( $\lambda$  and  $p$  are related via  $\lambda = \frac{p}{n+p}$ ),

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if  $K$  is an ellipsoid. For  $\lambda \in (-\infty, 0]$ , which corresponds to  $p \in (-n, 0]$ , we use Corollary 4 and get that for any  $C_2^+$  convex body  $K$ ,

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if  $K$  is an ellipsoid and if  $\lambda \geq 1$ , which corresponds to  $p \in [-\infty, -n)$ , then

$$c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B_2^n)},$$

where  $c$  is a universal constant. For  $p \geq 1$  these inequalities were proved by Lutwak [25] and for all other  $p$  by Werner and Ye [31].

Second, the functional definition  $as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)$  and  $as_p(K)$  may not coincide for  $p < 0$ . Indeed, if  $\partial K \setminus (\partial K)_+$  has non zero Lebesgue measure then  $as_p(K) = +\infty$  while it can happen that the corresponding functional definition is finite. The simplest example is the convex hull of the point  $(-e_1)$  with the half unit sphere  $\{\sum x_i^2 = 1, x_1 \geq 0\}$ .

Note that  $\left(\frac{\|\cdot\|_K^2}{2}\right)^* = \frac{\|\cdot\|_{K^\circ}^2}{2}$ , where  $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall x \in K\}$  is the polar body of  $K$ . Thus the functional duality relation (19) implies the identity

$$\forall \lambda \in \mathbb{R}, \quad as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) = as_{1-\lambda}\left(\frac{\|\cdot\|_{K^\circ}^2}{2}\right).$$

Together with Theorem 3 and taken  $\lambda = p/(n+p)$ , we get the classical duality relation

$$as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$$

for any  $p > 0$ . Moreover, this is also valid for any  $p \neq -n$  when  $(\partial K)_+$  has full measure in  $\partial K$ . This duality relation was proved in [19] for  $p > 0$  and for all  $p \neq -n$  in [31], with some more regularity assumption when  $p < 0$ .

## 4 The $L_p$ -affine surface area for $s$ -concave functions.

The purpose of this section is to generalize Definition 1, the functional version of  $L_p$ -affine surface area, to the context of  $s$ -concave functions for  $s > 0$ . We could have defined

$F_1(t) = F_2(t) = F^{(s)}(t) = (1 - st)_+^{1/s}$ , where  $a_+ = \max\{a, 0\}$ . Since  $F^{(s)}$  is log-concave and non-increasing, one has according to (11),  $F = F^{(s)}$  and when  $s \rightarrow 0$ , it recovers the previous case of  $F(t) = e^{-t}$ . However, when  $\psi$  is convex,  $F \circ \psi$  and  $F \circ \psi^*$  are not satisfying a good duality relation. Instead of the Legendre duality, we follow in this section another point of view, coming from the duality introduced in [2] for  $s$ -concave functions.

#### 4.1 The $s$ -concave duality.

We need few notations to explain the definition. Let  $s \in (0, +\infty)$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ . Following Borell [7], we say that  $f$  is  $s$ -concave if for every  $\lambda \in [0, 1]$  and all  $x$  and  $y$  such that  $f(x) > 0$  and  $f(y) > 0$ ,

$$f((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^s + \lambda f(y)^s)^{1/s}.$$

Since  $s > 0$ , it is equivalent to assuming that  $f^s$  is concave on its support. For the construction, we assume that  $f$  is upper semi-continuous. Let  $S_f$  be the convex set  $\{x : f(x) > 0\}$  and assume that 0 belongs to the interior of  $S_f$ . This can be done by choosing correctly the origin of the space  $\mathbb{R}^n$  and by assuming that  $f$  is not a trivial function. This will not affect the construction. We define the  $(s)$ -Legendre dual of  $f$  as

$$f_{(s)}^\circ(y) = \inf_{x \in S_f} \frac{(1 - s\langle x, y \rangle)_+^{1/s}}{f(x)}.$$

It coincides with the definition introduced in [2, 4]. Another point of view is to define a function  $\psi$  on  $S_f$  by

$$\psi(x) = \frac{1 - f^s(x)}{s}, \quad x \in S_f. \quad (23)$$

and to associate a new dual function  $\psi_{(s)}^*$  defined by

$$\psi_{(s)}^*(y) = \sup_{x \in S_f} \frac{\langle x, y \rangle - \psi(x)}{1 - s\psi(x)} \quad (24)$$

As  $f > 0$  on  $S_f$ ,  $\psi$  is well defined and since  $f$  is  $s$ -concave,  $\psi$  is convex on  $S_f$ . Observe that  $\psi < \frac{1}{s}$ , which means that  $1 - s\psi > 0$  on  $S_f$ . We can now define the  $(s)$ -Legendre dual of  $f$  as

$$f_{(s)}^\circ(y) = \left(1 - s\psi_{(s)}^*(y)\right)^{1/s}, \quad \forall y \in S_{f_{(s)}^\circ}$$

where  $S_{f_{(s)}^\circ} = \{y, 1 - s\psi_{(s)}^*(y) > 0\}$ . By definition,  $f_{(s)}^\circ$  is  $s$ -concave and upper semi-continuous. It is not difficult to see that as for the Legendre transform,  $(f_{(s)}^\circ)_{(s)}^\circ = f$  or equivalently that  $(\psi_{(s)}^*)_{(s)}^* = \psi$ . Moreover, it can be seen that for  $s > 0$ ,  $S_{f_{(s)}^\circ} = \frac{1}{s}S_f^\circ = \{z, \forall x \in S_f, \langle x, z \rangle < 1\}$ .

There is an implicit relation between the classical Legendre function  $\psi^*$  and the  $(s)$ -Legendre function  $\psi_{(s)}^*$  given by the formula

$$\forall y \in S_{f_{(s)}^\circ}, \left(1 - s\psi_{(s)}^*(y)\right) \left(1 + s\psi^*\left(\frac{y}{1 - s\psi_{(s)}^*(y)}\right)\right) = 1. \quad (25)$$

Our definition in the  $s$ -concave case is the following.

**Definition 2.** For any  $s > 0$ , let  $f$  be an  $s$ -concave function and  $\psi$  be the convex function associated above. For any  $\lambda \in \mathbb{R}$ , let

$$as_\lambda^{(s)}(\psi) = \frac{1}{1+ns} \int_{X_\psi} \frac{(1-s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)} (\det \nabla^2 \psi(x))^\lambda}{(1+s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{\lambda(n+\frac{1}{s}+1)-1}} dx.$$

It does not correspond to Definition 1 with particular function  $F_1$  and  $F_2$ . As in the log-concave case, we call it the  $L_\lambda$ -affine surface area of an  $s$ -concave function  $f$ . This is motivated by two main reasons. Like in Theorem 2, we prove in Theorem 4 a satisfactory duality relation, from which we deduce a reverse log-Sobolev inequality for  $s$ -concave measures. Moreover, in the case  $s = 1/k > 0$  where  $k$  is an integer, this functional affine surface area corresponds to an  $L_p$ -affine surface area of a convex body build apart from  $f$  in dimension  $n+k$ . Indeed, as in [2], we associate the convex body  $K_s(f)$  in  $\mathbb{R}^{n+\frac{1}{s}}$ ,

$$K_s(f) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^{\frac{1}{s}} : \frac{x}{\sqrt{s}} \in S_f, |y| \leq f^s \left( \frac{x}{\sqrt{s}} \right) \right\}.$$

Then the  $L_\lambda$ -affine surface area of  $f$  is the  $L_p$ -affine surface area of  $K_s(f)$  with  $p = (n + \frac{1}{s}) \frac{\lambda}{1-\lambda}$ ,

$$(1+ns) as_\lambda^{(s)}(\psi) = \frac{as_p((K_s(f)))}{s^{\frac{n}{2}} \text{vol}_{\frac{1}{s}-1} \left( S^{\frac{1}{s}-1} \right)}. \quad (26)$$

Identity (26) follows from Proposition 5 in [10].

Finally, we note that, as it is the case for log-concave functions, the  $L_\lambda$ -affine surface area for  $s$ -concave functions is also affine invariant under the action of  $SL_n$  and has a degree of homogeneity.

**Theorem 4.** Let  $f$  be an upper semi-continuous  $s$ -concave function with its corresponding convex function  $\psi$ . Assume that  $0 \in S_f$ . Let  $\lambda \in \mathbb{R}$  then

$$as_{1-\lambda}^{(s)}(\psi_{(s)}^\star) = as_\lambda^{(s)}(\psi).$$

*Proof.* Let us start with the case when  $f$  is sufficiently smooth, say  $f$  is twice continuously differentiable on  $S_f$  and its Hessian is non zero. Then  $\psi$  is  $\mathcal{C}_+^2$  on  $\Omega_\psi$  and

$$as_\lambda^{(s)}(\psi) = \frac{1}{1+ns} \int_{\Omega_\psi} \frac{(1-s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)} (\det \nabla^2 \psi(x))^\lambda}{(1+s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{\lambda(n+\frac{1}{s}+1)-1}} dx. \quad (27)$$

A simple computation tells that the supremum in (24) is attained at the point  $x \in S_f$  such that

$$y = \frac{1-s\langle x, y \rangle}{1-s\psi(x)} \nabla \psi(x) \text{ which means } y = (1-s\psi_{(s)}^\star(y)) \nabla \psi(x).$$

Therefore  $\langle x, y \rangle = \frac{1-s\langle x, y \rangle}{1-s\psi(x)} \langle x, \nabla \psi(x) \rangle$  and we get that

$$\frac{1}{1-s\psi_{(s)}^\star(y)} = \frac{1-s\psi(x)}{1-s\langle x, y \rangle} = 1 + s(\langle \nabla \psi(x), x \rangle - \psi(x)). \quad (28)$$



Finally, we get that

$$\psi_{(s)}^*(y) = \frac{\langle x, y \rangle - \psi(x)}{1 - s\psi(x)}$$

if and only if

$$y = \frac{\nabla\psi(x)}{1 + s(\langle \nabla\psi(x), x \rangle - \psi(x))} = \frac{\nabla\psi(x)}{1 + s\psi^*(\nabla\psi(x))}.$$

We define the change of variable

$$\frac{\nabla\psi(x)}{1 + s(\langle \nabla\psi(x), x \rangle - \psi(x))} = T_\psi(x). \quad (29)$$

A straightforward computation shows that

$$d_x T_\psi = \frac{1}{1 + s\psi^*(\nabla\psi(x))} \left( \text{Id} - \frac{s}{1 + s\psi^*(\nabla\psi(x))} x \otimes \nabla\psi(x) \right) \nabla^2\psi(x).$$

Since

$$\det \left( \text{Id} - \frac{s}{1 + s\psi^*(\nabla\psi(x))} x \otimes \nabla\psi(x) \right) = 1 - \frac{s}{1 + s\psi^*(\nabla\psi(x))} \langle x, \nabla\psi(x) \rangle$$

we get that the the Jacobian of  $T_\psi$  at  $x$  is given by

$$dy = |\det d_x T_\psi| dx = \frac{1 - s\psi(x)}{(1 + s(\langle \nabla\psi(x), x \rangle - \psi(x)))^{n+1}} \det \nabla^2\psi(x) dx. \quad (30)$$

As the the duality  $(\psi_{(s)}^*)_{(s)}^* = \psi$  holds, we see that  $T_\psi \circ T_{\psi_{(s)}^*} = \text{Id}$  and  $T_{\psi_{(s)}^*} \circ T_\psi = \text{Id}$  from which it is easy to deduce that for  $y = T_\psi(x)$ ,

$$\det (d_x T_\psi) \det (d_y T_{\psi_{(s)}^*}) = 1. \quad (31)$$

We make the change of variable  $y = T_\psi(x)$  in formula (27). From (28) and the fact that  $(\psi_{(s)}^*)_{(s)}^* = \psi$ , we have

$$\frac{1}{1 - s\psi_{(s)}^*(y)} = 1 + s(\langle \nabla\psi(x), x \rangle - \psi(x)) \text{ and } \frac{1}{1 - s\psi(x)} = 1 + s(\langle \nabla\psi_{(s)}^*(y), y \rangle - \psi_{(s)}^*(y)).$$

Combining with (30) and (31) we get that

$$\det \nabla^2\psi(x) \left( \frac{1 - s\psi_{(s)}^*(y)}{1 + s(\langle \nabla\psi_{(s)}^*(y), y \rangle - \psi_{(s)}^*(y))} \right)^{n+2} \det \nabla^2\psi_{(s)}^*(y) = 1. \quad (32)$$

Posing  $y = T_\psi(x)$  we get

$$\begin{aligned} (1 + ns) as_\lambda^{(s)}(\psi) &= \int_{\Omega_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)-1} (\det \nabla^2\psi(x))^{\lambda-1}}{(1 + s(\langle x, \nabla\psi(x) \rangle - \psi(x)))^{(\lambda-1)(n+1)+\frac{\lambda}{s}-1}} |\det d_x T_\psi| dx \\ &= \int_{\Omega_{\psi_{(s)}^*}} \frac{\left(1 - s\psi_{(s)}^*(y)\right)^{(n+2)(1-\lambda)+(\lambda-1)(n+1)+\frac{\lambda}{s}-1} \left(\det \nabla^2\psi_{(s)}^*(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi_{(s)}^*(y) \rangle - \psi_{(s)}^*(y))\right)^{(n+2)(1-\lambda)+\left(\frac{1}{s}-1\right)(1-\lambda)-1}} dy \\ &= \int_{\Omega_{\psi_{(s)}^*}} \frac{\left(1 - s\psi_{(s)}^*(y)\right)^{\lambda\left(\frac{\lambda}{s}-1\right)} \left(\det \nabla^2\psi_{(s)}^*(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi_{(s)}^*(y) \rangle - \psi_{(s)}^*(y))\right)^{(n+1+\frac{1}{s})(1-\lambda)-1}} dy \\ &= (1 + ns) as_{1-\lambda}^{(s)}(\psi_{(s)}^*). \end{aligned}$$

This concludes the proof in the smooth case. In the full generality, we need several observations. By (3), we have a.e. in  $\Omega_\psi$ ,

$$(1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x))) = 1 + s\psi^*(\nabla \psi(x))$$

Therefore, we can use a result of Mc Cann [26], see (10), to get

$$\begin{aligned} (1 + ns) \, as_\lambda^{(s)}(\psi) &= \int_{X_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)} (\det \nabla^2 \psi(x))^\lambda}{(1 + s\psi^*(\nabla \psi(x)))^{\lambda(n+\frac{1}{s}+1)-1}} dx \\ &= \int_{X_{\psi^*}} \frac{(1 - s\psi(\nabla \psi^*(z)))^{\left(\frac{1}{s}-1\right)(1-\lambda)} (\det \nabla^2 \psi^*(z))^{1-\lambda}}{(1 + s\psi^*(z))^{\lambda(n+\frac{1}{s}+1)-1}} dz. \end{aligned} \quad (33)$$

We make the change of variable  $z = T(y) = \frac{y}{1 - s\psi_{(s)}^*(y)}$ . Since  $1 - s\psi_{(s)}^*$  is convex, it is not difficult to see that  $T$  is an injective map. From (25), our change of variable is equivalent to  $y = \frac{z}{1 + s\psi^*(z)}$ . Therefore, a.e. in  $\Omega_{\psi_{(s)}^*}$ , a similar computation to (30) gives

$$|\det d_y T| = \frac{1 + s(\psi_{(s)}^*)^*(\nabla \psi_{(s)}^*(y))}{(1 - s\psi_{(s)}^*(y))^{n+1}}. \quad (34)$$

It can also be proved that it maps  $X_\psi$  to  $X_{\psi_{(s)}^*}$  and that the Alexandrov derivatives satisfy (this is similar to proposition A.1 in [26])

$$\left( \frac{1 - s\psi_{(s)}^*(y)}{1 + s(\psi_{(s)}^*)^*(\nabla \psi_{(s)}^*(y))} \right)^{n+2} \det \nabla^2 \psi_{(s)}^*(y) = \det \nabla^2 \psi^*(z). \quad (35)$$

Since  $(\psi_{(s)}^*)_{(s)}^* = \psi$ , we deduce from (25) that

$$\forall x \in S_f, (1 - s\psi(x)) \left( 1 + s(\psi_{(s)}^*)^* \left( \frac{x}{1 - s\psi(x)} \right) \right) = 1$$

Using (25) and the definition of  $T$ , it is not difficult to prove that a.e. in  $\Omega_{\psi^*}$ ,

$$\frac{\nabla \psi^*(z)}{1 - s\psi(\nabla \psi^*(z))} = \nabla \psi_{(s)}^*(y), \quad \text{for } z = Ty$$

which shows that for  $z = Ty$ ,

$$1 - s\psi(\nabla \psi^*(z)) = \frac{1}{1 + s(\psi_{(s)}^*)^*(\nabla \psi_{(s)}^*(y))}. \quad (36)$$

We have all the tools in hand to make the change of variable  $z = T(y)$  in (33) and to deduce from (34), (35), (36) that

$$(1 + ns) \, as_\lambda^{(s)}(\psi) = \int_{X_{\psi_{(s)}^*}} \frac{(1 - s\psi_{(s)}^*(y))^{\lambda\left(\frac{1}{s}-1\right)} (\det \nabla^2 \psi_{(s)}^*(y))^{1-\lambda}}{(1 + s(\langle y, \nabla \psi_{(s)}^*(y) \rangle - \psi_{(s)}^*(y)))^{\lambda(n+1+\frac{1}{s})(1-\lambda)-1}} dy.$$

This finishes the proof of the duality relation in the general case.  $\square$

## 4.2 Consequences of the duality relation

In this section, we suppose that  $f$  satisfies more regularity assumptions: it is twice continuously differentiable on  $S_f$ , its Hessian is non zero on  $S_f$ ,  $\lim_{x \rightarrow \partial S_f} f^s(x) = 0$  and recall that the origin belongs to the interior of  $S_f$ . With such assumptions,  $X_\psi = S_f$  and  $X_{\psi_{(s)}^*} = S_{f_{(s)}^\circ}$  and we remark that the definition of  $as_\lambda^{(s)}(\psi)$  is made in such a way that

$$as_0^{(s)}(\psi) = \int_{S_f} f(x) dx \quad \text{and} \quad as_1^{(s)}(\psi) = \int_{S_{f_{(s)}^\circ}} f_{(s)}^\circ(y) dy. \quad (37)$$

Indeed,

$$\begin{aligned} as_0^{(s)}(\psi) &= \frac{1}{1+ns} \int (1 - s\psi(x))^{\frac{1}{s}-1} (1 + s(\langle \nabla \psi(x), x \rangle - \psi(x))) dx \\ &= \frac{1}{1+ns} \int f(x) \left( 1 - s \frac{\langle \nabla f(x), x \rangle}{f(x)} \right) dx = \int f(x) dx, \end{aligned}$$

where the last equality follows from Stokes formula and the fact that  $\lim_{x \rightarrow \partial S_f} f^s(x) = 0$ . The second relation follows from the duality relation proved in Theorem 4.

In a way similar to the proof of Corollary 1 and Theorem 1, it is possible to deduce from Theorem 4 some isoperimetric inequalities and a general reverse log-Sobolev inequality in the  $s$ -concave setting.

**Proposition 1.** *Let  $f$  be a  $s$ -concave function that satisfies the regularity assumption defined at the beginning of Section 4.2 and  $\psi$  be its associated convex function. Then*

$$\begin{aligned} \forall \lambda \in [0, 1], \quad as_\lambda^{(s)}(\psi) &\leq \left( \int_{\mathbb{R}^n} f dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} f_{(s)}^\circ dx \right)^\lambda; \\ \forall \lambda \notin [0, 1], \quad as_\lambda^{(s)}(\psi) &\geq \left( \int_{\mathbb{R}^n} f dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} f_{(s)}^\circ dx \right)^\lambda. \end{aligned}$$

*Proof.* We use Hölder inequality, (37) to prove the first inequality.

$$\begin{aligned} as_\lambda^{(s)}(\psi) &\leq \frac{1}{1+ns} \left[ \left( \int_{\mathbb{R}^n} (1 - s\psi(x))^{\frac{1}{s}-1} (1 - s\psi(x) + s\langle x, \nabla \psi(x) \rangle) dx \right)^{1-\lambda} \right. \\ &\quad \left. \left( \int_{\mathbb{R}^n} \frac{\det \nabla^2 \psi(x)}{(1 - s\psi(x) + s\langle x, \nabla \psi(x) \rangle)^{n+\frac{1}{s}}} dx \right)^\lambda \right] \\ &= \left( \int_{\mathbb{R}^n} f dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} f_{(s)}^\circ dx \right)^\lambda. \end{aligned}$$

Similarly, we use reverse Hölder inequality to prove the second inequality.  $\square$

The next theorem gives the log-Sobolev inequality for  $s$ -concave functions. There, we put

$$d\mu = (1 - s\psi)^{\left(\frac{1}{s}-1\right)} (1 + s(\langle \nabla \psi, x \rangle - \psi)) \frac{dx}{1+ns}.$$

By (37),  $\mu$  is a probability measure on  $\mathbb{R}^n$ . We let  $S(\mu) = \int -\log\left(\frac{d\mu}{dx}\right) d\mu$  be the Shannon entropy of  $\mu$ .

**Theorem 5.** *Let  $f$  be an  $s$ -concave function that satisfies the regularity assumption defined at the beginning of Section 4.2 and  $\psi$  be its associated convex function. Assume moreover that  $f$  is even and that  $\int f(x)dx = 1$ . Then*

$$\begin{aligned} \int \log(\det(\nabla^2 \psi(x))) d\mu &\leq \int \log\left((1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{\frac{1}{s}+n}\right) d\mu - S(\mu) \\ &\quad + \log\left(\left(\frac{\pi}{s}\right)^n \frac{(1 + ns) \left(\Gamma(1 + \frac{1}{2s})\right)^2}{\left(\Gamma(1 + \frac{n}{2} + \frac{1}{2s})\right)^2}\right). \end{aligned} \quad (38)$$

There is equality if and only if there is a positive definite matrix  $A$  such that  $f(x) = c_0 \left(1 - s|Ax|^2\right)^{\frac{1}{2s}}$ , where  $c_0 = \left(\frac{\pi}{s}\right)^{-\frac{n}{2}} \left(\frac{\Gamma(1+\frac{1}{2s})}{\Gamma(1+\frac{n}{2}+\frac{1}{2s})}\right)^{-1}$ .

**Remark.**  $S(\gamma_n) = \log(2\pi e)^{\frac{n}{2}}$ . Therefore, the right hand side the inequality (38) tends to  $2[S(\gamma_n) - S(\mu)]$  for  $s \rightarrow 0$  and we recover the inequality of Theorem 1.

*Proof.* The proof follows the line of the proof of Theorem 1 presented in Section 2. By the definition (24) of  $\psi_{(s)}^*$ , we have for all  $x \in S_f$  and for all  $y \in \frac{S_f^\circ}{s}$  that

$$f(x)f_{(s)}^\circ(y) = (1 - s\psi(x))^{\frac{1}{s}} (1 - s\psi_{(s)}^*(y))^{\frac{1}{s}} \leq (1 - s\langle x, y \rangle)^{\frac{1}{s}}.$$

We let  $\rho(t) = (1 - st)_+^{\frac{1}{2s}}$ . As  $f \equiv 0$  outside  $S_f$  and  $f_{(s)}^\circ \equiv 0$  outside  $\frac{S_f^\circ}{s}$ , the functions  $f$  and  $f_{(s)}^\circ$  satisfy the assumption (12) with  $z_0 = 0$  because  $f$  is even. It follows from (13) that

$$\left(\int f dx\right) \left(\int f_{(s)}^\circ dx\right) \leq \left(\int (1 - s|x|^2)_+^{\frac{1}{2s}} dx\right)^2 = \left(\frac{\pi}{s}\right)^n \frac{\left(\Gamma(1 + \frac{1}{2s})\right)^2}{\left(\Gamma(1 + \frac{n}{2} + \frac{1}{2s})\right)^2}. \quad (39)$$

By Theorem 4, we have  $\int f_{(s)}^\circ = as_0^{(s)}(\psi_{(s)}^*) = as_1^{(s)}(\psi)$  which means that

$$\begin{aligned} \int f_{(s)}^\circ &= \frac{1}{1 + ns} \int_{X_\psi} \frac{\det \nabla^2 \psi(x)}{(1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{(n+\frac{1}{s})}} dx \\ &= \frac{1}{1 + ns} \int_{X_\psi} \frac{\det \nabla^2 \psi(x)}{(1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{(n+\frac{1}{s})}} \frac{dx}{d\mu(x)} d\mu(x) \end{aligned}$$

Since  $\int f = 1$ ,  $\mu$  is a probability measure and we get from Jensen inequality

$$\begin{aligned} \log\left(\int f_{(s)}^\circ\right) &\geq S(\mu) - \log(1 + ns) + \int \log(\det \nabla^2 \psi) d\mu \\ &\quad - \int \log\left((1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{\frac{1}{s}+n}\right) d\mu. \end{aligned}$$

Therefore, with (39) and as  $\int f dx = 1$ ,

$$\begin{aligned} \int \log(\det(\nabla^2 \psi)) d\mu &\leq \int \log\left((1 + s(\langle x, \nabla \psi(x) \rangle - \psi(x)))^{\frac{1}{s}+n}\right) d\mu - S(\mu) \\ &\quad + \log(1 + ns) + \log\left(\left(\frac{\pi}{s}\right)^n \frac{\left(\Gamma(1 + \frac{1}{2s})\right)^2}{\left(\Gamma(1 + \frac{n}{2} + \frac{1}{2s})\right)^2}\right). \end{aligned}$$

When equality holds in (38), then in particular equality holds in the Blaschke Santaló inequality (39). It was proved in [16] that this happens if and only if, in our situation,  $f(x) = c_0 \left(1 - s |Ax|^2\right)^{\frac{1}{2s}}$ , for a positive definite matrix  $A$  and where  $c_0$  as above is chosen such that  $\int f dx = 1$ . On the other hand, it is easy to see that equality holds in (38), when  $f(x) = c \left(1 - s |Ax|^2\right)^{\frac{1}{2s}}$ , for a positive definite matrix  $A$  and a positive constant  $c$ .  $\square$

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